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# Optimal Reduced-Order Estimators for Parameter-Dependent Systems

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**The optimal reduced-order estimation problem in which the plant model depends on parameters, which are known at the time of operation, is considered. Such cases occur when the parameters are either measurable or their changing values are known in advance. A method for approximation of the updated estimator, without complete re-solution of the problem, is given. A similar approach is used to develop a new algorithm for the numerical solution of the nominal reduced-order estimation problem.**

## Nomenclature

| | |
|---|---|
| $A, B, C$ | = $n \times n$, $n \times m$, $r \times n$ system matrices |
| $A_e, B_e, C_e, D_e$ | = $n_e \times n_e$, $n_e \times r$, $l \times n_e$, $l \times r$ estimator matrices |
| $\mathrm{Diag}(Z)$ | = diagonal of matrix $Z$ |
| $E$ | = expected value |
| $I$ | = identity matrix |
| $L$ | = $l \times n$ matrix |
| $Q, \hat{Q}, \hat{P}$ | = $n \times n$ nonnegative definite matrices |
| $Q_a$ | = $n \times m$ matrix |
| $R$ | = $n \times n$ positive-definite matrix |
| $R^{s \times p}$ | = $s \times p$ real matrices |
| $V_1, V_2$ | = $m \times m$, $r \times r$ positive-definite covariance matrices |
| $V_{12}$ | = $m \times r$ cross-covariance matrix |
| $\mathrm{Vec}(Z)$ | = columns of matrix $Z$ stacked in a vector |
| $w_1, w_2$ | = $m$-, $r$-dimensional white noises |
| $\boldsymbol{x}, \boldsymbol{x}_e, \boldsymbol{y}, \boldsymbol{y}_e$ | = $n$-, $n_e$-, $r$-, $l$-dimensional vectors |
| $\Gamma, G$ | = $n_e \times n$ matrices |
| $\tau, \tau_\perp$ | = $n \times n$ projection matrices |

## I. Introduction

APPROXIMATION of high-order, complex systems by lower-order, relatively simpler ones is one of the fundamental problems in linear system theory and has received considerable attention for many years. Order reduction can be applied to models of the plant, feedback controllers, and state estimators. Among these three types of dynamic systems the need for order reduction is perhaps most obvious in state estimation because models are usually used in off-line applications, and controllers, unless designed by some automatic method such as linear-quadratic-Gaussian (LQG) $H_\infty$, etc., need not have high order. Standard state estimators, on the other hand, are on-line devices whose order is equal to that of the system.

One suboptimal approach to the design of reduced-order estimators is to use a reduced-order model, obtained by one of the existing methods such as balanced realization,[1] Hankel norm approximation,[2] or $L_2$ optimization,[3,4] and then design a full-order estimator for it. Another approach is to first design the full-order estimator and then treat it as a model and reduce its order by use of the same methods. However, none of these approaches is optimal because order reduction and optimization are not independent of each other. Direct design methods, e.g., Refs. 5–7, combine order reduction and estimation within a single framework. The optimal reduced-order estimation problem was first solved by Bernstein and Hyland.[8] Since then it has been extended to several directions, such as singular cases,[9] unstable plants,[10] observer–estimator structure,[11,12] and steady-state preservation.[13]

The starting point of all these methods is a given, constant plant model. In many cases the plant model depends on physical parameters that may change. If those changes are unknown then one should use either robust estimation methods[14,15] or adaptive estimation.[16] Robust estimation means that a single constant estimator is designed to have the required performance for a family of plants. In adaptive estimation the parameters are identified in real time and the estimator is updated accordingly. However, in some applications the parameters can be assumed to be known at any time, either because they are measurable, e.g., altitude, or because one has prior knowledge regarding the changing values. The question is how to update the optimal reduced-order estimator with the known values of the parameters. This problem is essentially gain scheduling for this special type of estimator. Because estimation is an open-loop procedure and both the plant and the estimator (for all values of the parameters) are stable, no stability problem arises, contrary to gain scheduling in control.[17]

The physical situation considered in this work is that the parameter is a constant that may change and is not a continuous function of time. Therefore the optimal steady-state estimator is considered. If the parameters change continuously but slowly enough, then the steady-state estimator for their current value can still be used as it represents an adiabatic approximation.[18] In the full-order case, updating the estimator whenever the parameters change requires the solution of the algebraic Riccati equation, which is sometimes possible during operation. Because the computation effort required for the nominal solution of the reduced-order estimator is much higher, it cannot be done in real time and a different approach should be taken.

In this paper an approximated updating scheme, based on a series expansion of the optimal reduced-order estimator, is presented. Most of the calculation is done off line, which makes it suitable for on-line applications. A similar result for updating reduced-order models was considered in Ref. 19. However, as was discussed in the preceding paragraphs, order reduction and estimation are coupled and the problem needs separate treatment.

The derivation leads also to a new method of numerical solution of the nominal reduced-order estimation problem. The idea is to define a system fictitiously with a known reduced-order estimator as the nominal system and to regard the passage to the true system as a parameter change. The method can be an alternative to homotopic methods,[20,21] which are commonly used to solve this problem.

The material is organized as follows. Section II presents the optimal reduced-order estimation problem and its solution and discusses the numerical solution of the equations. In Sec. III an updating scheme, which is the main result of the paper, is derived. The applicability of the method is demonstrated by means of an example in Sec. III. Section V deals with the numerical solution for the nominal case based on the model updating schemes of Sec. III. The results are summarized and discussed in Sec. VI.

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## II. Optimal Reduced-Order Estimation

Given the stable $n$th-order system

$$\dot{x}(t) = Ax(t) + Bw_1(t) \tag{1}$$

$$y(t) = Cx(t) + w_2(t) \tag{2}$$

where

$$E\left\{\begin{bmatrix} w_1(t) \\ w_2(t)\end{bmatrix}\begin{bmatrix} w_1^T(\tau) & w_2^T(\tau)\end{bmatrix}\right\} = \begin{bmatrix} V_1 & V_{12} \\ V_{12}^T & V_2\end{bmatrix}\delta(t-\tau)$$

$$V_2 > 0 \tag{3}$$

it is desired to find an $n_e$th-order estimator ($n_e < n$) given by

$$\dot{x}_e(t) = A_e x_e(t) + B_e y(t) \tag{4}$$

$$y_e(t) = C_e x_e(t) + D_e y(t) \tag{5}$$

that minimizes

$$J = \lim_{t\to\infty} E\left\{[Lx(t) - y_e(t)]^T R[Lx(t) - y_e(t)]\right\} \tag{6}$$

The matrix $L$ determines the linear combinations of the state variables that are to be estimated, and $R > 0$ is a constant weighting matrix. It is well known that the full-order optimal estimator, i.e., the Kalman filter, is independent of $L$ and $R$, but this is not the case in reduced-order optimization.

The following Lemma is required for the solution of the problem.

*Lemma 2.1 (Ref. 8):* Let $\hat{Q}$, $\hat{P} \in R^{n\times n}$ be nonnegative definite and rank $\hat{Q} =$ rank $\hat{P} =$ rank $\hat{Q}\hat{P} = n_e$. Then there exist (nonunique) $G$, $\Gamma \in R^{n_e\times n}$ and nonsingular $M \in R^{n_e\times n_e}$ such that $\hat{Q}\hat{P} = \Gamma M G^T$ and $\Gamma G^T = I_{n_e}$. The projections $\tau = G^T\Gamma$ and $\tau_\perp = I_n - \tau$ are independent of the particular choice of $G$ and $\Gamma$.

The optimal reduced-order estimator is given by the following theorem.

*Theorem 2.1 (Ref. 8):* Suppose that $(A_e, B_e, C_e)$ solves the optimal reduced-order estimation problem. Then it is given by

$$\left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e\end{array}\right] = \left[\begin{array}{c|c} \Gamma\left(A - Q_a V_2^{-1} C\right)G^T & \Gamma Q_a V_2^{-1} \\ \hline LG^T & 0\end{array}\right] \tag{7}$$

where

$$Q_a = QC^T + BV_{12} \tag{8}$$

$\Gamma$, $G^T$, $\tau$, and $\tau_\perp$ are as in Lemma 2.1 and the nonnegative definite matrices $\hat{Q}$, $\hat{P}$, $Q \in R^{n\times n}$ satisfy

$$AQ + QA^T + BV_1B^T - Q_aV_2^{-1}Q_a^T + \tau_\perp Q_aV_2^{-1}Q_a^T\tau_\perp^T = 0 \tag{9}$$

$$A\hat{Q} + \hat{Q}A^T + Q_aV_2^{-1}Q_a^T - \tau_\perp Q_aV_2^{-1}Q_a^T\tau_\perp^T = 0 \tag{10}$$

$$\left(A - Q_aV_2^{-1}C\right)^T\hat{P} + \hat{P}\left(A - Q_aV_2^{-1}C\right) + L^TRL - \tau_\perp^T L^TRL\tau_\perp = 0 \tag{11}$$

$$\text{rank}(\hat{Q}) = \text{rank}(\hat{P}) = \text{rank}(\hat{Q}\hat{P}) = n_e \tag{12}$$

Equations (9–12) are independent of the specific realization of $(A_e, B_e, C_e)$ and have a distinct structure. On the other hand, these equations are not in a convenient form for numerical solution, as the rank condition is hard to enforce. To obtain a more suitable form we introduce in the following Lemma a specific factorization.

*Lemma 2.2 (Ref. 20):* Suppose that $\hat{Q}$ and $\hat{P} \in R^{n\times n}$ are nonnegative definite and condition (12) is satisfied for some $n_e < n$. Then there exist $G_0$, $\Gamma_0 \in R^{n_e\times n}$ and diagonal positive-definite $\Lambda_0 \in R^{n_e\times n_e}$ such that

$$\hat{Q} = G_0^T\Lambda_0G_0, \qquad \hat{P} = \Gamma_0^T\Lambda_0\Gamma_0, \qquad \Gamma_0G_0^T = I_{n_e}$$

First we note that $\hat{Q}\hat{P} = G_0^T\Lambda_0^2\Gamma_0$; hence the triplet $(G_0, \Gamma_0, \Lambda_0^2)$ is a factorization of $\hat{Q}\hat{P}$ as in Lemma 2.1 and represents a specific realization of the estimator. If the diagonal entries of $\Lambda_0$ are distinct and arranged in a certain order, e.g., descending, then this factorization is unique up to multiplication of the $i$th row of both $\Gamma_0$ and $G_0$ by $-1$. Hence $\Gamma_0$, $G_0$, and $\Lambda_0$, together with $Q$, constitute an appropriate set of unknowns for the solution.

By premultiplying Eq. (10) by $\Gamma_0$, postmultiplying Eq. (11) by $G_0^T$, and using the relationships $\Gamma_0\tau_\perp = 0$ and $\tau_\perp G_0^T = 0$, we obtain

$$AQ + QA^T + BV_1B^T - Q_aV_2^{-1}Q_a^T + \left(I_n - G_0^T\Gamma_0\right)Q_aV_2^{-1}Q_a^T\left(I_n - G_0^T\Gamma_0\right)^T = 0 \tag{13}$$

$$\Gamma_0\left(AG_0^T\Lambda_0G_0 + G_0^T\Lambda_0G_0A^T + Q_aV_2^{-1}Q_a^T\right) = 0 \tag{14}$$

$$\left[\left(A - Q_aV_2^{-1}C\right)^T\Gamma_0^T\Lambda_0\Gamma_0 + \Gamma_0^T\Lambda_0\Gamma_0\left(A - Q_aV_2^{-1}C\right) + L^TRL\right]G_0^T = 0 \tag{15}$$

$$\Gamma_0G_0^T - I_{n_e} = 0 \tag{16}$$

The only obstacle that remains, except for the actual solution of this set of coupled nonlinear equations, is that there are more equations than unknowns. Equations (13–16) contain $n^2 + 2n \times n_e + n_e^2$ scalar equations but only $n^2 + 2n \times n_e + n_e$ unknowns. However, it was shown in Ref. 19 that Eqs. (14) and (15) contain each only $n_e \times n + (n_e^2 - n_e)/2$ independent equations, which makes the number of independent equations and unknowns equal. Equations (13–16) are the basis for the main results of the paper in Sec. III. We now drop the subscript 0 from $\Gamma$, $\Lambda$, and $G^T$, but they still represent the values obtained by the factorization in Lemma 2.2.

## III. Parameter-Dependent Reduced-Order Estimator

Suppose that the system depends on a vector of known parameters $\alpha$, i.e.,

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)w_1(t) \tag{17}$$

$$y(t) = C(\alpha)x(t) + w_2(t) \tag{18}$$

Then the optimal reduced-order estimator also depends on $\alpha$:

$$\left[\begin{array}{c|c} A_e(\alpha) & B_e(\alpha) \\ \hline C_e(\alpha) & D_e(\alpha)\end{array}\right] = \left[\begin{array}{c|c} \Gamma(\alpha)\left[A(\alpha) - Q_a(\alpha)V_2^{-1}C(\alpha)\right]G^T(\alpha) & \Gamma(\alpha)Q_a(\alpha)V_2^{-1} \\ \hline LG^T(\alpha) & 0\end{array}\right] \tag{19}$$

The dependence of $A$, $B$, and $C$ is explicit whereas that of $\Gamma$, $G$, and $Q$ is implicit by means of Eqs. (13–16). The ability to update the parameters during operation is often a necessity, as explained in the introduction, and an on-line solution of Eqs. (13–16) is unrealistic. In this section we derive two approximated updating schemes of the reduced-order estimator for a given change of the parameter. Both are based on power-series expansion about the nominal $\alpha^0$. For the sake of simplicity of the presentation we assume for the moment that $\alpha$ is a scalar. The generalization for the vector case is given in Remark 3.4.

*Scheme I:* In this approach $\Gamma(\alpha)$, $G(\alpha)$, and $Q(\alpha)$ are approximated by a truncated Taylor series:

$$\hat{\Gamma}(\alpha) = \Gamma(\alpha^0) + \frac{\partial\Gamma}{\partial\alpha}(\alpha - \alpha^0) + \cdots + \frac{1}{p!}\frac{\partial^p\Gamma}{\partial\alpha^p}(\alpha - \alpha^0)^p \tag{20}$$

$$\hat{G}(\alpha) = G(\alpha^0) + \frac{\partial G}{\partial\alpha}(\alpha - \alpha^0) + \cdots + \frac{1}{p!}\frac{\partial^pG}{\partial\alpha^p}(\alpha - \alpha^0)^p \tag{21}$$

$$Q(\alpha) = Q(\alpha^0) + \frac{\partial Q}{\partial\alpha}(\alpha - \alpha^0) + \cdots + \frac{1}{p!}\frac{\partial^pQ}{\partial\alpha^p}(\alpha - \alpha^0)^p \tag{22}$$

All the derivatives are evaluated at $\alpha = \alpha^0$ and this is the convention from now on. The approximations are then used in Eq. (19) instead of the exact $\Gamma(\alpha)$, $G(\alpha)$, and $Q(\alpha)$:

$$\left[\begin{array}{c|c} A_e(\alpha) & B_e(\alpha) \\ \hline C_e(\alpha) & D_e(\alpha) \end{array}\right] \cong \left[\begin{array}{c|c} \hat{\Gamma}(\alpha)\big[A(\alpha) - \hat{Q}_a(\alpha)V_2^{-1}C(\alpha)\big]\hat{G}^T(\alpha) & \hat{\Gamma}(\alpha)\hat{Q}_a(\alpha)V_2^{-1} \\ \hline L\hat{G}^T(\alpha) & 0 \end{array}\right] \tag{23}$$

*Scheme II:* In this approach the estimator matrices are approximated directly:

$$\hat{A}_e(\alpha) = A_e(\alpha^0) + \frac{\partial A_e}{\partial \alpha}(\alpha - \alpha^0) + \cdots + \frac{1}{p!}\frac{\partial^p A_e}{\partial \alpha^p}(\alpha - \alpha^0)^p \tag{24}$$

$$\hat{B}_e(\alpha) = B_e(\alpha^0) + \frac{\partial B_e}{\partial \alpha}(\alpha - \alpha^0) + \cdots + \frac{1}{p!}\frac{\partial^p B_e}{\partial \alpha^p}(\alpha - \alpha^0)^p \tag{25}$$

$$\hat{C}_e(\alpha) = C_e(\alpha^0) + \frac{\partial C_e}{\partial \alpha}(\alpha - \alpha^0) + \cdots + \frac{1}{p!}\frac{\partial^p C_e}{\partial \alpha^p}(\alpha - \alpha^0)^p \tag{26}$$

The off-line calculations of both schemes are almost identical because the partial derivatives of the estimator matrices are calculated from those of $\Gamma$, $G$, and $Q$. From an on-line operation point of view, Scheme II is superior because it involves less calculation and is in a lower dimension then Scheme I. Our experience, based on several examples, shows that the accuracy of both schemes for the same order of approximation $p$ is comparable. We therefore proceed with Scheme II; yet the derivation includes the required steps for Scheme I as well. Differentiating Eq. (7) with respect to $\alpha$ yields

$$\frac{\partial A_e}{\partial \alpha} = \frac{\partial \Gamma}{\partial \alpha}\bar{A}G^T + \Gamma\frac{\partial \bar{A}}{\partial \alpha}G^T + \Gamma\bar{A}\frac{\partial G^T}{\partial \alpha} \tag{27}$$

$$\frac{\partial B_e}{\partial \alpha} = \frac{\partial \Gamma}{\partial \alpha}B + \Gamma\frac{\partial B}{\partial \alpha} \tag{28}$$

$$\frac{\partial C_e}{\partial \alpha} = \frac{\partial C}{\partial \alpha}G^T + C\frac{\partial G^T}{\partial \alpha} \tag{29}$$

where

$$\bar{A} \triangleq A - Q_a V_2^{-1} C \tag{30}$$

$$\frac{\partial \bar{A}}{\partial \alpha} = \frac{\partial A}{\partial \alpha} - \frac{\partial Q_a}{\partial \alpha}V_2^{-1}C - Q_a V_2^{-1}\frac{\partial C}{\partial \alpha} \tag{31}$$

$$\frac{\partial Q_a}{\partial \alpha} = \frac{\partial B}{\partial \alpha}V_{12} + \frac{\partial Q}{\partial \alpha}C^T + Q\frac{\partial C^T}{\partial \alpha} \tag{32}$$

By substituting Eqs. (30–32) into Eq. (27), we obtain

$$\frac{\partial A_e}{\partial \alpha} = f_1\Bigg(\underbrace{A, B, C, \frac{\partial A}{\partial \alpha}, \frac{\partial B}{\partial \alpha}, \frac{\partial C}{\partial \alpha}, \Gamma, G, Q,}_{\text{known}} \frac{\partial \Gamma}{\partial \alpha}, \frac{\partial G}{\partial \alpha}, \frac{\partial Q}{\partial \alpha}\Bigg) \tag{33}$$

and similar expressions for $B_e$ and $C_e$. $\partial A/\partial\alpha$, $\partial B/\partial\alpha$, and $\partial C/\partial\alpha$, which appear in Eq. (33), are known matrices, and therefore we need to calculate only $\partial Q/\partial\alpha$, $\partial\Gamma/\partial\alpha$, and $\partial G^T/\partial\alpha$. At this point it should be noted that the fact that $\Gamma$ and $G^T$ in Eqs. (13–16) represent a specific realization is an advantage because the changes that are required correspond to that realization.

Differentiating Eqs. (13–16), which can be symbolically written as

$$H(A, B, C, \Gamma, G, \Lambda, Q) = 0 \tag{34}$$

leads to

$$\frac{\partial H}{\partial \alpha} = F_1\Bigg(\underbrace{A, B, C, \frac{\partial A}{\partial \alpha}, \frac{\partial B}{\partial \alpha}, \frac{\partial C}{\partial \alpha}, \Gamma, G, \Lambda, Q,}_{\text{coefficients}} \underbrace{\frac{\partial \Gamma}{\partial \alpha}, \frac{\partial G}{\partial \alpha}, \frac{\partial \Lambda}{\partial \alpha}, \frac{\partial Q}{\partial \alpha}}_{\text{unknowns}}\Bigg) = 0 \tag{35}$$

These equations are linear in the unknowns and can be easily solved. Substituting the solutions into Eq. (33), we get the first-order term in the series expansion. To obtain a higher-order term we proceed in a similar manner and differentiate Eq. (35) once more. $\partial Q/\partial\alpha$, $\partial\Gamma/\partial\alpha$, and $\partial G^T/\partial\alpha$ have already been calculated and are known at this stage, so the only unknowns are the second-order derivatives of $\Gamma$, $G$, and $Q$, which will be used in the evaluation of the second-order derivatives of $A_e$, $B_e$, and $C_e$. Repeating the same procedure $k$ times, we get the following results:

$$\frac{\partial^k A_e}{\partial \alpha^k} = \sum_{i=0}^{k}\sum_{j=0}^{k}\beta_{ij}^k\frac{\partial^i \Gamma}{\partial \alpha^i}\frac{\partial^j \bar{A}}{\partial \alpha^j}\frac{\partial^{k-i-j}G^T}{\partial \alpha^{k-i-j}} \tag{36}$$

$$\frac{\partial^k B_e}{\partial \alpha^k} = \sum_{i=0}^{k}\gamma_i^k\frac{\partial^i \Gamma}{\partial \alpha^i}\frac{\partial^{k-i}Q_a}{\partial \alpha^{k-i}}V_2^{-1} \tag{37}$$

$$\frac{\partial^k C_e}{\partial \alpha^k} = L\frac{\partial^k G^T}{\partial \alpha^k} \tag{38}$$

where

$$\frac{\partial^k \bar{A}}{\partial \alpha^k} = \frac{\partial^k A}{\partial \alpha^k} - \sum_{i=0}^{k}\gamma_i^k\frac{\partial^i Q_a}{\partial \alpha^i}V_2^{-1}\frac{\partial^{k-i}C}{\partial \alpha^{k-i}} \tag{39}$$

$$\frac{\partial^k Q_a}{\partial \alpha^k} = \frac{\partial^k B}{\partial \alpha^k}V_{12} + \sum_{i=0}^{k}\gamma_i^k\frac{\partial^i Q}{\partial \alpha^i}\frac{\partial^{k-i}C^T}{\partial \alpha^{k-i}} \tag{40}$$

and

$$\beta_{ij}^k = \begin{cases} \dfrac{k!}{i!j!(k-i-j)!} & \text{if} \quad i+j \le k \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

$$\gamma_i^k = \frac{k!}{i!(k-i)!} \tag{42}$$

To calculate these values we need the derivatives of $\Gamma$, $G$, and $Q$ up to $k$th order. The results are summarized in the following lemma, in which we use the notation $X_k = \partial^k X/\partial\alpha^k$, where $X$ is any variable.

*Lemma 3.1:* $Q_k$, $\Gamma_k$, $G_k$, and $\Lambda_k$ are given, for all $k$, by

$$\bar{A}Q_k + Q_k\bar{A}^T - \big(G_k^T\Gamma + G^T\Gamma_k\big)Q_a\tau_\perp^T - \tau_\perp Q_a\big(\Gamma^T G_k + \Gamma_k^T G\big) + \tau_\perp\big(Q_k C^T V_2^{-1}Q_a^T + Q_a V_2^{-1}CQ_k\big)\tau_\perp^T = D^{1,k} \tag{43}$$

$$\Gamma_k E + \Gamma A G_k^T\Lambda G + \Gamma A G^T\Lambda_k G + \Gamma A G^T\Lambda G_k + \Gamma G_k^T\Lambda G A^T + \Gamma G^T\Lambda_k G A^T + \Gamma G^T\Lambda G_k A^T + \Gamma Q_k C^T V_2^{-1}Q_a^T + \Gamma Q_a V_2^{-1}CQ_k = D^{2,k} \tag{44}$$

$$FG_k^T - C^T V_2^{-1}CQ_k\hat{P}G^T - \hat{P}Q_k C^T V_2^{-1}CG^T + \bar{A}^T\Gamma_k^T\Lambda + \bar{A}^T\Gamma^T\Lambda_k + \bar{A}^T\Gamma^T\Lambda\Gamma_k G^T + \Gamma_k^T\Lambda\Gamma A G^T + \Gamma^T\Lambda_k\Gamma A G^T + \Gamma^T\Lambda\Gamma_k A G^T = D^{3,k} \tag{45}$$

$$\Gamma_k G^T + \Gamma G_k^T = D^{4,k} \tag{46}$$

where

$$E \triangleq A\hat{Q} + \hat{Q}A^T + Q_a V_2^{-1} Q_a^T \tag{47}$$

$$F \triangleq \bar{A}^T \hat{P} + \hat{P}\bar{A} + L^T R L \tag{48}$$

The $D^{j,k}$ consist of $Q$, $\Gamma$, $G^T$, and $\Lambda$ and their $k-1$ first derivatives. The expressions for them are fairly long and are therefore given in the Appendix.

*Proof:* For $k = 1$, Eqs. (43–46) are the detailed version of the conceptual equation (35). We obtain them by differentiating Eqs. (13–16) with respect to $\alpha$ and moving all the known terms, which are given in the Appendix by setting $k = 1$, to the right-hand side. When Eqs. (43–46) are differentiated and only the highest-order derivatives $Q_k$, $\Gamma_k$, $G_k^T$, and $\Lambda_k$ are kept on the left-hand side, the structure remains the same and only the right-hand side changes. □

*Remark 3.1*: Equations (43–46) are linear in $Q_k$, $\Gamma_k$, $G_k^T$, and $\Lambda_k$, where the right-hand sides, though looking very complex, consist of known quantities at the $k$th step. By means of Kronecker products and a similar operation for diagonal matrices, these equations can be written in the standard form,

$$\begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{Vec}\left(G_k^T\right) \\ \mathrm{Vec}(\Gamma_k) \\ \mathrm{Diag}(\Lambda_k) \\ \mathrm{Vec}(Q_k) \end{bmatrix} = \begin{bmatrix} \mathrm{Vec}(D^{1,k}) \\ \mathrm{Vec}(D^{2,k}) \\ \mathrm{Vec}(D^{3,k}) \\ \mathrm{Vec}(D^{4,k}) \end{bmatrix} \tag{49}$$

where $\mathrm{Vec}(\cdot)$ is a vector obtained when the columns of $(\cdot)$ are stacked.

*Remark 3.2*: Equation (33) has $n^2 + 2nn_e + n_e^2$ rows (equations) and $n^2 + 2nn_e + n_e$ unknowns. However, the partial derivatives are evaluated at a point that is an exact solution of Eqs. (13–16); hence Eq. (49) is consistent. One can either use a left inverse of the coefficient matrix or simply delete $n_e^2 - n_e$ equations.

*Remark 3.3*: An important property of the solution is that the coefficient matrix is independent of the derivatives order $k$, and only the right-hand side of Eq. (49) changes. Hence the procedure involves only one matrix inversion (or an equivalent operation on the coefficient matrix) and the amount of computation is only weakly affected by the order of the approximation, i.e., the number of terms used in the series. This is very favorable from a computation point of view.

*Remark 3.4*: There is no conceptual change in the results when the system depends on $N$ parameters $\alpha_i$ instead of a single one. The following changes are made in the entire derivation:

$$\frac{\partial^p X}{\partial \alpha^p} \to \sum_{p_1} \cdots \sum_{p_N} \frac{\partial^p X}{\partial \alpha_1^{p_1} \cdots \partial \alpha_N^{p_N}}, \qquad \sum_{i=1}^{N} p_i = p \tag{50}$$

$$\frac{\partial^p X}{\partial \alpha^p}(\alpha - \alpha^0)^p \to \sum_{p_1} \cdots \sum_{p_N} \frac{\partial^p X}{\partial \alpha_1^{p_1} \cdots \partial \alpha_N^{p_N}} \times \left(\alpha_1 - \alpha_1^0\right)^{p_1} \cdots \left(\alpha_N - \alpha_N^0\right)^{p_N} \tag{51}$$

To be more specific, Eqs. (27–35) remain the same with $\alpha_i$ instead of $\alpha$ and they are calculated $N$ times. Hence in first-order approximation the changes are minimal. In higher-order approximations the unknowns in Eqs. (43–46) represent an increase in the order of the derivative of a single parameter, say $\alpha_j$:

$$Q_k \to Q_{k_1,\ldots,k_{j-1},k_j+1,k_{j+1},\ldots,k_N}, \qquad \sum_{i=1}^{N} k_i = k - 1 \tag{52}$$

The left-hand side of Eqs. (43–46) or, equivalently, Eq. (49), is constant for each $\alpha_i$. The matrix inversion is performed not once, as in Remark 3.3, but $N$ times. Going back from the derivatives of the projection to those of the estimator matrices in Eqs. (36–38) is straightforward yet involves summation over all the parameters.

The procedure for obtaining the coefficient matrices for the parameter-dependent reduced-order estimator is summarized in the following algorithms for both schemes.

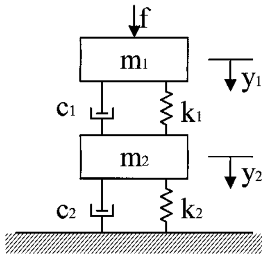


**Fig. 1 System of the example.**

*Scheme I:*
*Step 0:* Set $k = 1$.
*Step 1:* Calculate $D^{j,k}$, the right-hand-side terms in Eq. (49).
*Step 2:* Solve Eq. (49) for $Q_k$, $\Gamma_k$, $G_k$, and $\Lambda_k$.
*Step 3:* If $k = p$, stop. Otherwise set $k = k + 1$ and go to step 1.
The outputs are the constant matrices $Q_k$, $\Gamma_k$, and $G_k$, $k = 1, \ldots, p$. The updated $(\Gamma, G^T, Q)$ are calculated on line from Eqs. (20–22) and the updated $(A_e, B_e, C_e)$ follow from Eq. (19).

*Scheme II:*
*Step 0:* Set $k = 1$.
*Step 1:* Calculate $D^{j,k}$, the right-hand-side terms in Eq. (49).
*Step 2:* Solve Eq. (49) for $Q_k$, $\Gamma_k$, $G_k$, and $\Lambda_k$.
*Step 3:* Calculate $\partial^k A_e/\partial\alpha^k$, $\partial^k B_e/\partial\alpha^k$, and $\partial^k C_e/\partial\alpha^k$.
*Step 4:* If $k = p$, stop. Otherwise set $k = k + 1$ and go to step 1.
The outputs are the constant matrices $\partial^k A_e/\partial\alpha^k$, $\partial^k B_e/\partial\alpha^k$, and $\partial^k C_e/\partial\alpha^k$, $k = 1, \ldots, p$. The updated $(A_e, B_e, C_e)$ are calculated on line from Eqs. (24–26), a procedure that involves only multiplication of given matrices by scalars and one summation.

## IV. Example

Consider the system in Fig. 1, which can be regarded as a simplified model of a flexible structure such as an airplane wing, a lightweight robot arm, etc. The random force $f$ is modeled as white noise with intensity $V_1$. The output is defined as the displacement $y_1$, and the measurement noise is $w_2$.

The state-space model is given by

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1}{m_1} & \dfrac{k_1}{m_1} & -\dfrac{c_1}{m_1} & \dfrac{c_1}{m_1} \\ \dfrac{k_1}{m_2} & -\dfrac{k_1+k_2}{m_2} & \dfrac{c_1}{m_2} & -\dfrac{c_1+c_2}{m_2} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_1} \\ 0 \end{bmatrix} f \tag{53}$$

$$\boldsymbol{y} = [1 \quad 0 \quad 0 \quad 0]\boldsymbol{x} + w_2 \tag{54}$$

The nominal values are

$$m_1 = m_2 = 1, \qquad k_1 = k_2 = 1, \qquad c_1 = 0.1, \qquad c_2 = 0.3$$

$$V_1 = 0.75, \qquad V_2 = 0.1, \qquad V_{12} = 0$$

$L$ and $R$ were both selected as $I_4$, i.e., the entire state vector is estimated. $k_1$ may change and is given by

$$k_1 = 1 + \Delta k \tag{55}$$

The goal is to find an estimator of the order of 2 for this fourth-order system for a range of $\Delta k$. Applying the method described in Sec. III for second-order approximation, we get

$$A_e = \begin{bmatrix} -0.4702 & 0.6451 \\ -2.1010 & -1.0093 \end{bmatrix} + \begin{bmatrix} 0.0594 & 0.0990 \\ 0.4358 & 0.3608 \end{bmatrix} \Delta k + \begin{bmatrix} -0.0471 & -0.0708 \\ -0.6422 & -0.5498 \end{bmatrix} \Delta k^2 \tag{56}$$

$$B_e = \begin{bmatrix} -0.3545 \\ -1.2717 \end{bmatrix} + \begin{bmatrix} -0.0940 \\ -0.0953 \end{bmatrix} \Delta k + \begin{bmatrix} 0.0576 \\ -0.1286 \end{bmatrix} \Delta k^2 \tag{57}$$

$$C_e = \begin{bmatrix} -1.3198 & -0.7415 \\ -0.8358 & -0.3444 \\ 0.3243 & -1.0860 \\ 0.1415 & 0.7842 \end{bmatrix} + \begin{bmatrix} 0.5236 & 0.3581 \\ -0.0483 & -0.0218 \\ -0.0638 & 0.4595 \\ 0.0444 & -0.0726 \end{bmatrix} \Delta k + \begin{bmatrix} -0.4875 & -0.3644 \\ 0.1119 & -0.0511 \\ 0.0312 & -0.4491 \\ -0.0030 & 0.1616 \end{bmatrix} \Delta k^2 \tag{58}$$

The optimal reduced-order estimator for a given $k_1$ is always stable. Because Eq. (56) is an approximation of the true $A_e$, one should check the range of $\Delta k$ for which it is stable. A simple search reveals that $A_e$ in Eq. (56) is stable for $-3.7 < \Delta k < 8.6$, which is well beyond the physical range. As the criterion of accuracy we use the normalized deviation of the approximated estimator cost $J_{\text{apr}}$ from the cost $J_{\text{ex}}$ for the exact solution (which was calculated separately for each value of $k_1$):

$$\varepsilon = \frac{J_{\text{apr}} - J_{\text{ex}}}{J_{\text{ex}}} \tag{59}$$

Figure 2 shows $\varepsilon$ as a function of $\Delta k$ for approximations of the order of 2, 1, and 0 (which means that the estimator remains unchanged). The approximations are very good even for 50% deviation from the nominal value, and, as was expected, the accuracy increases with the order of the approximation. For $\Delta k = -50\%$ the deviation in the cost $\varepsilon$ without updating is $\sim$80%. The first-order approximation improves it to 13%, and second-order approximation improves it to 2%.

## V. Application to the Solution of the Nominal Problem

By changing our point of view, we can use the method that was developed in Sec. III to solve the nominal problem that is given in Theorem 2.1. Let

$$\bar{P} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \tag{60}$$

be the system for which we want to find the optimal $n_e$th-order estimator and

$$P_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & 0 \end{bmatrix} \tag{61}$$

be a system for which the solution is given, i.e., $\Gamma$, $G$, and $Q$ are known. Then we can define

$$P(\alpha) = P_0 + \alpha(\bar{P} - P_0) \tag{62}$$

and clearly

$$P(1) = \bar{P} \tag{63}$$

Hence we can find the reduced-order estimator for $\bar{P}$ by using the methods of Sec. III with $\alpha = 1$. Assuming that the series converges, then by adding more and more terms we get closer to the exact solution. Note that, although in general $\alpha = 1$ is not regarded as small, it is the product $\alpha(\bar{P} - P_0)$ that should be relatively small because the derivatives of $Q, \Gamma, G$, and $\Lambda$ are proportional to $\partial P / \partial \alpha$, i.e., $\bar{P} - P_0$. Nevertheless, unless one starts close enough to $\bar{P}$, many terms must be used. Note that the formulas for high-order derivatives for the case in which the system depends linearly on a single parameter are simpler because second- and higher-order derivatives of $(A, B, C)$ are zero.

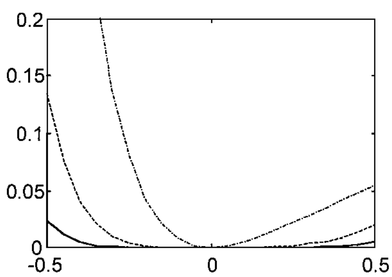


**Fig. 2 Normalized error $\varepsilon$ vs the relative change in $k_1$: - · -, order 0; - - -, order 1; ——, order 2.**

As was already mentioned, the coefficients on the left-hand side of the equations are the same and the procedure involves therefore only one matrix inversion. Equation (49) is solved recursively to obtain additional terms in the series. It remains to show how to get the system $P_0$ with its known reduced-order estimator. The simplest way is to start with a system whose full-order optimal estimator is nonminimal and whose McMillan degree is $n_e$. This happens if the dimension of the controllable subspace of $(A_0, B_0)$ is $n_e$. A natural choice is

$$A_0 = A, \qquad C_0 = C, \qquad B_0 = V\beta \tag{64}$$

where $V \in R^{n \cdot n_e}$ consists of $n_e$ eigenvectors of $A$ and $\beta$ is arbitrary. To make $P_0$ close to $\bar{P}$ we choose

$$\beta = V^{+} B \tag{65}$$

because it minimizes $\|B - V\beta\|_F$. This approach compares well with homotopic methods for the solution of the optimal projection equations that characterize the optimal reduced-order estimator. The key point is that the method requires only one inversion of a coefficient matrix, whereas in homotopic and Newton–Raphson methods it is done every step. The result is a considerable reduction in computational effort. A less favorable characteristic is that in the recursive algorithm derived in Sec. III each stage requires the solutions of all previous stages, which adds up to considerable memory requirements.

## VI. Summary

Methods for obtaining a reduced-order estimator that depends on varying, yet known, parameters of the system were developed. Two schemes for updating the optimal estimator without re-solution of the problem were suggested. Both were derived by repeated differentiation of the equations that characterize the optimal reduced-order estimator. The on-line calculations consist of only evaluating a matrix polynomial in the deviation of the parameter from its nominal value. The required off-line calculations are only weakly affected by the level of the approximation, as the same system of linear equations is solved at each step. The same results were used to derive a method for numerical solution of the nominal case. The solution is expressed as a deviation from a fictitious nominal system with a known reduced-order estimator. Such systems can be found by considering a nonminimal system that is close to the true system. A method of constructing such a system is given.

## Appendix: Formulas for $D^{j,k}$

$$D^{1,k} = \hat{D}^{1,k} + \hat{D}^{1,k^T} + \sum_{i=1}^{k-1} \gamma_i^k \Big( Q_{a_i} V_2^{-1} Q_{a_{k-i}}^T + \tau_\perp Q_{a_i} V_2^{-1} Q_{a_{k-i}}^T \tau_\perp^T \Big) - \sum_{i=0}^{k} \gamma_i^k B_i V_1 B_{k-i}^T - \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \beta_{ij}^k \tau_{\perp i} Q_{Vj} \tau_{\perp k-i-j}^T \tag{A1}$$

$$D^{2,k} = \Gamma\Big(\hat{D}^{2,k} + \hat{D}^{2,k^T}\Big) - \sum_{i=1}^{k-1} \gamma_i^k \Gamma_i E_{k-i} \tag{A2}$$

$$D^{3,k} = \Big(\hat{D}^{3,k} + \hat{D}^{3,k^T}\Big) G^T - \sum_{i=1}^{k-1} \gamma_i^k F_i G_{k-i}^T \tag{A3}$$

$$D^{4,k} = -\sum_{i=1}^{k-1} \gamma_i^k \Gamma_{k-i} G_i^T \tag{A4}$$

where

$$\hat{D}^{1,k} = -\left(A_k - Q_a V_2^{-1} C_k\right) Q + B_k V_{12} V_2^{-1} Q_a^T - \tau_\perp \left(Q C_k^T + B_k V_{12}\right) V_2^{-1} Q_a^T \tau_\perp^T - \sum_{i=1}^{k-1} \gamma_i^k \Big[\left(A_i - Q_a V_2^{-1} C_i\right) \times Q_{k-i} - G_i^T \Gamma_{k-i} Q_V \tau_\perp^T + \tau_\perp Q_i C_{k-i}^T V_2^{-1} Q_a^T \tau_\perp^T\Big] \tag{A5}$$

$$\hat{D}^{2,k} = A_k \hat{Q} + \left(Q C_k^T + B_k V_{12}\right) V_2^{-1} Q_a^T + A Q^{ij} + \sum_{i=1}^{k-1} \gamma_i^k \left(A_i \hat{Q}_{k-i} + Q_i C_{k-i}^T V_2^{-1} Q_a^T + \frac{1}{2} Q_{a_i} V_2^{-1} Q_{a_{k-i}}^T\right) \tag{A6}$$

$$\hat{D}^{3,k} = -\left(A_k^T - C^T V_2^{-1} C_k Q - C_k^T V_2^{-1} Q_a^T - C^T V_2^{-1} V_{12}^T B_k^T\right) \hat{P} - A^T P^{ij} + \sum_{i=1}^{k-1} \gamma_i^k \left(C^T V_2^{-1} C_{k-i} Q_i \hat{P} + C_{k-i}^T V_2^{-1} Q_{a_i}^T \hat{P} + \bar{A}_i^T \hat{P}_{k-i}\right) \tag{A7}$$

$$Q_V \triangleq Q_a V_2^{-1} Q_a^T, \qquad Q_{Vk} = \sum_{i=0}^{k} \gamma_i^k Q_{a_i} V_2^{-1} Q_{a_{k-i}}^T \tag{A8}$$

$$\tau_{\perp k} = -\sum_{i=0}^{k} \gamma_i^k G_i^T \Gamma_{k-i} \tag{A9}$$

$$E_k = \sum_{i=0}^{k} \gamma_i^k \left(A_i \hat{Q}_{k-i} + \hat{Q}_i A_{k-i}^T + Q_{a_i} V_2^{-1} Q_{a_{k-i}}^T\right) \tag{A10}$$

$$\hat{Q}_k = G_k^T \Lambda G + G^T \Lambda_k G + G^T \Lambda G_k + Q^{ij} \tag{A11}$$

$$Q^{ij} = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \beta_{ij}^k G_i^T \Lambda_j G_{k-i-j} \tag{A12}$$

$$F_k = \sum_{i=0}^{k} \gamma_i^k \left(\bar{A}_i^T \hat{P}_{k-i} + \hat{P}_i \bar{A}_{k-i}\right) \tag{A13}$$

$$\hat{P}_k = \Gamma_k^T \Lambda \Gamma + \Gamma^T \Lambda_k \Gamma + \Gamma^T \Lambda \Gamma_k + P^{ij} \tag{A14}$$

$$P^{ij} = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \beta_{ij}^k \Gamma_i^T \Lambda_j \Gamma_{k-i-j} \tag{A15}$$

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